CHAPTER NINE

Frequency Response Methods

9.1 Introduction
It was pointed earlier that in practice the performance of a feedback control system is more preferably measured by its time - domain response characteristics. This is in contrast to the analysis and design of systems in the communication field, where the frequency response is of more importance, since in this case most of the signals to be processed are either sinusoidal or periodic in nature. However, analytically, the time response of a control system is usually difficult to determine, especially in the case of high - order systems. In the design aspects, there are no unified ways of arriving at a design system given the time - domain specifications, such as peak overshoot, rise time, delay time and setting time. On the other hand, there is a wealth of graphical methods available in the frequency - domain analysis, all suitable for the analysis and design of linear feedback control system.

Once the analysis and design are carried out in the frequency - domain, the time - domain behavior of the system can be interpreted based on the relationships that exist between the time - domain and the frequency - domain properties. Therefore, might consider that the main purpose of conducting control systems analysis and design in the frequency domain is merely to use the techniques as a convenient vehicle toward the same objectives as with time - domain method.

Frequency response methods were developed in 1930s and 1940s by Nyquist, Bode Nichols, and many others. The frequency response methods are most powerful in conventional control theory. They are also indispensable to robust control theory.

The principle of frequency response testing is based in comparing the input and output of the elements under test over a wide range of frequencies, when subjected to a sinusoidal input signal.

The Nyquist stability criterion enables to investigate both the absolute and relative stabilities of linear closed loop systems from knowledge of their open loop frequency response characteristics. An advantage of the frequency response approach is that frequency response tests are; in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment. Often the transfer functions of complicated components can be determined experimentally by frequency response tests. In addition, the frequency response approach has the advantages that a system may be designed so that the effects of undesirable noise are negligible and that such analysis and design can be extended to certain nonlinear control systems.
Although the frequency response of a control system presents a qualitative picture of the transient response, the correlation between frequency and transient responses is indirect, except for the case of second-order systems. In designing a closed-loop system, we adjust the frequency response characteristic of the open-loop transfer function by using several design criteria in order to obtain acceptable transient response characteristics of the system.

9.2 Frequency response coverage the following plots

9.2.1 Polar plot
Polar plot is a graph of \( \text{Im} [GH] \) versus \( \text{Re} [GH] \) on the \( [GH(jw)] \) - plane for \(-\infty < w < \infty\).

![Figure 9.1 Polar Plot](image)

9.2.2 Magnitude - Phase plot
The magnitude in decibel versus the phase on rectangular coordinates with \( w \) as a variable parameter is called Magnitude versus Phase Plot.

![Figure 9.2 Magnitude versus Phase](image)
9.2.3 Bode plot
The plot of the magnitude in decibel versus \( w \) (or \( \log w \)) in semilog (or rectangular) coordinate called Bode plot (corner plot).

\[
\begin{align*}
\left| G(s) H(s) \right| & = 10^{\frac{1}{20} \log_{10} |G(j \omega)|} \\
\angle G(s) H(s) & = \angle G(j \omega) + \angle H(j \omega)
\end{align*}
\]

Figure 9.3 Bode Plot

9.3 Frequency Response Characteristic
For closed loop control system
\[
M = \frac{G(s)}{1 + G(s) H(s)}
\]

\[let \quad s = j \omega\]
\[
M(j \omega) = \frac{G(j \omega)}{1 + G(j \omega) H(j \omega)}
\]

\[
= \left| \frac{G(j \omega)}{1 + G(j \omega) H(j \omega)} \right| M(j \omega)
\]

\[
magnitude \quad \text{phase angle}
\]

\(M(j \omega)\) may be regarded as the magnification of the feedback control system. A typical magnification curve of a feedback control system is shown in the following figure:
Mp - Resonance Peak: is the maximum value of the magnitude of the closed loop frequency response.

wp - Resonance frequency: is the frequency at which Mp occurs.

B.W - Band Width; is the range of frequencies (of the input) over which the system will respond satisfactorily. Satisfactorily will be at value equal to 0.707 of magnification.

wc - Cut off frequency: it is occurs when the magnitude ratio tend to 0.707 of its value.

9.4 Polar Plots (Nyquist Plot)

The polar plot of a sinusoidal transfer function G(jw) is a plot of the magnitude of G(jw) versus the phase angle of G(jw) on polar coordinates as w is varied from zero to infinity. Thus, the polar plot is the locus of vectors |G(jw)⟩∥G(jw)⟩ as w is varied from zero to infinity.

Note that in polar plots a positive (negative) phase angle is measured counter clockwise (clockwise) from positive real axis.

The polar plot is often called Nyquist plot. An example of such a plot is shown in Figure 9.4. Each point on the polar plot of G(jw) represents the terminal point of a vector at a particular value of w. in the polar plot, it is important to show the frequency graduation of the locus. The projections of G(jw) on the real and imaginary axes is real and imaginary components.

An advantage in using a polar plot is that it depicts the frequency response characteristics of a system over the entire frequency range in a single plot. One disadvantage is that the plot does not clearly indicate the contributions of each individual factor of the open loop transfer function.
Integrated and Derivative Factors $(jw)^{±1}$

The polar plot of $G(jw) = 1/ jw$ is the negative imaginary axis since

$$G(jw) = \frac{1}{jw} = -j \frac{1}{w} = \frac{1}{w} -90^\circ$$

The polar plot of $G(jw) = jw$ is the positive imaginary axis.

First order Factors $(1 + jw \ T)^{±1}$

For the sinusoidal transfer function

$$G(jw) = \frac{1}{1 + jw \ T} = \frac{1}{\sqrt{1 + w^2 T^2}} \left[-\tan^{-1} w \ T\right]$$

The values of $G(jw)$ at $w = 0$ and $w = 1/T$ are respectively

$$G(0) = 1[0^\circ] \quad \text{and} \quad G\left(\frac{1}{T}\right) = \frac{1}{\sqrt{2}}[-45^\circ]$$

If $w$ approaches infinity, the magnitude of $G(jw)$ approaches zero and the phase angle approaches $-90^\circ$. The polar plot of this transfer function is a semicircle as the frequency $w$ is varied from zero to infinity.
**Quadratic Factors** \( [1 + 2 \zeta \left( \frac{jw}{w_n} \right) + \left( \frac{jw}{w_n} \right)^2]^{\pm 1} \)

The low and high frequency portions of the polar plot the following sinusoidal transfer function

\[
G(jw) = \frac{1}{1 + 2 \zeta \left( \frac{jw}{w_n} \right) + \left( \frac{jw}{w_n} \right)^2} \quad \text{for} \quad \zeta > 0
\]

are given respectively by

\[
\lim_{w \to \infty} G(jw) = 110^\circ \quad \text{and} \quad \lim_{w \to \infty} G(jw) = 0180^\circ
\]

The polar plot of this sinusoidal transfer function starts at \( 110^\circ \) and ends at \( 0180^\circ \) as \( w \) increases from zero to infinity. Thus, the high frequency portion of \( G(jw) \) is tangent to the negative real axis.

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**General Shapes of Polar Plots**

The polar plots of a transfer function of the form

\[
G(jw) = \frac{K (1 + jwT_a)(1 + jwT_b) \ldots}{(jw)^2 (1 + jwT_i)(1 + jwT_j) \ldots}
\]

Where \( n > m \) or the degree of the denominator polynomial is greater than that of the numerator, will have the following general shapes:

1. **Type zero control systems, \( \lambda = 0 \):**

   The starting point of the polar plot (which correspond to \( w = 0 \)) is finite and is on the positive real axis. The tangent to the polar plot at \( w = 0 \) is perpendicular to the real axis. The terminal point, which corresponds to \( w = \infty \), is at the origin, and the curve is tangent to one of the axis.

   **Example:** \( G(jw) = \frac{K}{(1 + jwT_1)(1 + jwT_2)} \)

   at \( w = 0 \), \( G(jw) = K[0^\circ] \)

   \( w \to \infty \), \( G(jw) = 0[-180^\circ] \), \( \tan^{-1}\infty = 90^\circ \)

   Each factor of denominator contributes an angle of -90\(^\circ\) or 90\(^\circ\) in clockwise direction.
2. Type 1 Control System $\lambda = 1$:

The $jw$ term in the denominator contributes $-90$ to the total phase angle of $G(jw)$ for $0 \leq w \leq \infty$. At $w = 0$, the magnitude of $G(jw)$ is infinity, and the phase angle becomes $-90$. At low frequencies, the polar plot is asymptotic to a line parallel to the negative imaginary axis. At $w = \infty$, the magnitude becomes zero, and the curve converges to the origin and is tangent to one of the axes.

Example:

$$G(jw) = \frac{K}{(jw)(1+jwT_1)(1+jwT_2)(1+jwT_3)}$$

at $w \to 0$, $G(jw) = \infty \leftarrow 90^\circ$

at $w \to \infty$, $G(jw) = 0 \leftarrow 360^\circ$

$V_x$ is asymptotes as $w$ approaches to zero, $G(jw)$ approaches to infinity along to asymptotes $V_x$, and $V_x$ is found from:

$$V_x = \lim_{w \to 0} \text{Re}[G(jw)]$$

and $w_x$ is determined from

$\text{Im}[G(jw)] = 0$
3. Type 2 Control System $\lambda=2$:

The $(jw)^2$ term in the denominator contributes -180 to the total phase angle of $G(jw)$ for $0 \leq w \leq \infty$. At $w = 0$, the magnitude of $G(jw)$ is infinity, and the phase angle is equal to -180. At low frequencies, the polar plot is asymptotic to a line parallel to the negative real axis. At $w = \infty$, the magnitude becomes zero, and the curve is tangent to one of the axes.

Example: $G(jw) = \frac{K}{(jw)^2(1+jwT_1)(1+jwT_2)}$

- at $w \to 0$, $G(jw) = \infty[-180^\circ]$
- $w \to \infty$, $G(jw) = 0[-360^\circ]$
Rough Sketch of the Polar Plot

The sketching of the polar plot is facilitated by the following information:
1. The behavior of the magnitude and the phase at \( w = 0 \) and at \( w = \infty \).
2. The point of intersection of the polar plot with the real and imaginary axes, and the value of \( w \) at the intersections.

Absolute and Relative Stability

The simplified Nyquist criterion for system stability may be stated as follows:
If \( GH(jw) \) does not have poles in the right half \( s \)-plane, the closed loop system is stable, if and only if the -1 pint lies to the left of the polar plot when moving along this plot in the direction of increasing \( w \). That is, the polar plot passes on the right side of -1.

Gain Margin (GM)

GM is defined as the reciprocal of the magnitude of the open loop transfer function when the phase shift is 180. This is, therefore the factor by which the gain must be increased in order to produce instability.

Phase Margin (\( \Phi_{PM} \))

\( \Phi_{PM} \) is defined as 180 minus the phase angle of the open loop transfer function \([GH]\) of the frequency when the gain is unity. This is therefore the amount of the phase angle, would have to be increased to make the system unstable.

9.5 Bode Diagrams (or Logarithmic Plots)

A Bode diagram (plot) consists of two graphs: once is a plot of the logarithm of the magnitude of a sinusoidal transfer function; the other is a plot of the phase angle; both are plotted against the frequency on a logarithmic scale.

The standard representation of the logarithmic magnitude of \( G(jw) \) is \( 20\log |G(jw)| \), where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel, usually abbreviated dB. In the logarithmic representation, the curves are drawn on semi log paper, using the log scale for frequency and the linear scale for either magnitude (but in decibels) or phase angle (in degrees). The frequency range of interest determines the number of logarithmic cycles required on the abscissa.

The main advantage of using Bode diagram is that multiplication of the magnitudes can be converted into addition. Further more; a simple method for sketching an approximate log-magnitude curve is available. It is based on asymptotic approximations. Such approximation
by straight-line asymptotes is sufficient if only rough information on the frequency -response characteristics is needed. Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots. Expanding the low - frequency range by use of a logarithmic scale for the frequency is highly advantageous since characteristics at low frequencies are most important impractical systems. Although it is not possible to plot the curves right down to zero frequency because of the logarithmic frequency (log 0 = - ∞).

Note that the experimental determination of a transfer function can be made simple if frequency - response data are presented in the form of a Bode diagram.

**Basic Factors of G(jw)H(jw)**

The basic factors that vary frequently occur in arbitrary transfer function G(jw)H(jw) are:

1. Gain K.
2. Integrated and derivative factors (jw)^±1.
3. First - order factors (1 + jw T)^±1.
4. Quadratic factors \[(1 + 2ζ \left( \frac{jw}{w_n} \right) + \left( \frac{jw}{w_n} \right)^2)^{±1} \].

Once we become familiar with the logarithmic plots of these basic factors, it is possible to utilize them in constructing a composite logarithmic plot for any general form of G(jw)H(jw) by sketching for each factor and adding individual curves graphically; because adding the logarithms of the gains corresponds to multiplying them together.

**The Gain K**

A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value. The log - magnitude curve for a constant gain K is a horizontal straight line at the magnitude of 20 log K decibels. The phase angle of the gain K is zero. The effect of varying the gain K in the transfer function is that it raises or lowers the log - magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.
Integral factor $\frac{1}{jw}$

The logarithmic magnitude of $1/jw$ in decibel is

$$20 \log |\frac{1}{jw}| = -20 \log w \, dB$$

This is a straight line of slope -20 dB/decade. The phase angle of $1/jw$ is constant and equal to -90°.

Derivative factor $jw$

The logarithmic magnitude of $jw$ in decibel is

$$20 \log |jw| = 20 \log w \, dB$$

This is a straight line of slope 20 dB/decade. The phase angle of $jw$ is constant and equal to 90°.

First order factor

a. Factor $\frac{1}{1 + jwT}$

Log magnitude is

$$20 \log \left| \frac{1}{1 + jwT} \right| = -20 \log \sqrt{1+w^2T^2} \, dB$$

For low frequencies such that $w \ll 1/T$, the log magnitude may be approximated by

$$-20 \log \sqrt{1+w^2T^2} \approx -20 \log 1 = 0$$

Thus, the log magnitude curve at low frequencies is the constant 0 dB line. For high frequencies, such that $w \gg 1/T$,

$$-20 \log \sqrt{1+w^2T^2} \approx -20 \log wT \, dB$$

This is an approximate expression for the high frequency range. At $w = 1/T$, the log magnitude equals 0 dB; at $w = 10/T$, the log magnitude is -20 dB, the value of $-20 \log wT$.
db decreases by 20 db for every decade of w. for w \approx 1/T, the log magnitude curve is thus a straight line with a slope of -20 dB/decade.

The frequency at which the two asymptotes meet is called the corner frequency or break frequency. For the factor 1/(1 + jw T), the frequency w = 1/T is the corner frequency since at w = 1/T the two asymptotes have the same value. The corner frequency divides the frequency response curve into two regions; a curve for the low frequency region and a curve for high frequency region. The corner frequency is very important in sketching logarithmic frequency response curves.

The exact phase angle Φ of the factor 1/(1 + jw T) is

\[ \Phi = -\tan^{-1} wT \]

At zero frequency (w = 0), the phase angle is 0°. At corner frequency (w = 1/T), the phase angle is \[ \Phi = -\tan^{-1} \frac{1}{T} \] \(= -\tan^{-1} 1 = -45°\). At infinity (w → ∞), the phase angle becomes -90°. Since the phase angle is given by an inverse tangent function, the phase angle is skew symmetric about the inflection point at Φ = -45°.

b. Factor 1 + jw T

Log magnitude is 20 \log |1 + jwT| = 20 \log \sqrt{1 + w^2 T^{-2}} dB

For low frequencies such that w \ll 1/T, the log magnitude may be approximated by 20 \log \sqrt{1 + w^2 T^{-2}} \approx -20 \log 1 = 0

Thus, the log magnitude curve at low frequencies is the constant 0 dB line. For high frequencies, such that w \gg 1/T,

20 \log \sqrt{1 + w^2 T^{-2}} \approx 20 \log wT dB

The exact phase angle Φ of the factor (1 + jw T) is

\[ \Phi = \tan^{-1} wT \]

At zero frequency (w = 0), the phase angle is 0°. At corner frequency (w = 1/T), the phase angle is \[ \Phi = \tan^{-1} \frac{1}{T} = \tan^{-1} 1 = 45°\]. At infinity (w → ∞), the phase angle becomes 90°. Since the phase angle is given by an inverse tangent function, the phase angle is skew symmetric about the inflection point at Φ = 45°.
**Quadratic Factors**

**a. The factor**

\[
\frac{1}{1 + 2\zeta j\frac{w}{w_n} + (j\frac{w}{w_n})^2}
\]

If \(\zeta > 1\), this quadratic factor can be expressed as a product of two first order factors with real poles. If \(0 < \zeta < 1\), this quadratic factor is the product of two complex conjugate factors. Asymptotic approximations to the frequency response curves are not accurate for a factor with low values of \(\zeta\). This is because the magnitude and phase of the quadratic factor depend on both the corner frequency and the damping ratio \(\zeta\).

The asymptotic frequency-response curve may be obtained as follows. Since

\[
20 \log \left| \frac{1}{1 + 2\zeta j\frac{w}{w_n} + (j\frac{w}{w_n})^2} \right| = -20 \log \sqrt{(1 - \frac{w^2}{w_n^2})^2 + (2\zeta \frac{w}{w_n})^2}
\]

For low frequencies such that \(w \ll w_n\), the log magnitude becomes

\[-20 \log 1 = 0 \text{ dB}\]

The low-frequency asymptote is thus a horizontal line at 0 dB. For high frequencies such that \(w \gg w_n\), the log magnitude becomes

\[-20 \log \frac{w^2}{w_n^2} = -40 \log \frac{w}{w_n} \text{ dB}\]

The equation for the high-frequency asymptote is a straight line having the slope \(-40 \text{ dB/decade}\) since

\[-40 \log \frac{10w}{w_n} = -40 - 40 \log \frac{w}{w_n}\]

The high frequency asymptote intersects the low-frequency one at \(w = w_n\) since at this frequency

\[-40 \log \frac{w_n}{w_n} = -40 \log 1 = 0 \text{ dB}\]

This frequency, \(w_n\), is the corner frequency for the quadratic factor considered.

The phase angle of the quadratic factor \(\Phi = \tan^{-1} \left( \frac{2\zeta \frac{w}{w_n}}{1 - \frac{w^2}{w_n^2}} \right)\) is

\[
\frac{1}{1 + 2\zeta(j\frac{w}{w_n}) + (j\frac{w}{w_n})^2}
\]

The phase angle is a function of both \(w\) and \(\zeta\). At \(w = 0\), the phase angle equals 0°. At the corner frequency \(w = w_n\), the phase angle is -90° regardless of \(\zeta\), since
\[ \Phi = -\tan^{-1}\left(\frac{2\zeta}{\omega}\right) = -\tan^{-1}\infty = -90^\circ \]

At \( w = \infty \), the phase angle becomes \( -180^\circ \). The phase angle curve is skew symmetric about the inflection point (the point where \( \Phi = -90^\circ \)).
9.6 The Resonant Frequency, $w_r$, and the Resonant Peak Value, $M_r$

The magnitude of

$$G(jw) = \frac{1}{1 + 2\zeta j \frac{w}{w_n} + (j \frac{w}{w_n})^2}$$

is

$$|G(jw)| = \frac{1}{\sqrt{(1 - \frac{w_n^2}{w^2})^2 + (2\zeta \frac{w}{w_n})^2}}$$

If $|G(jw)|$ has a peak value at some frequency, this frequency is called the resonant frequency. Since the numerator of $|G(jw)|$ is constant, a peak value of $|G(jw)|$ will occur when

$$g(jw) = (1 - \frac{w_n^2}{w^2})^2 + (2\zeta \frac{w}{w_n})^2$$

is a minimum, and this equation can be written as

$$g(jw) = \left[\frac{w_n^2 - w_n^2(1 - 2\zeta^2)}{w_n^2}\right]^2 + 4\zeta^2(1 - \zeta^2)$$

is minimum value of $g(w)$ occur at

$$w = w_n \sqrt{1 - 2\zeta^2}$$

Thus the resonant frequency $w_r$ is

$$w_r = w_n \sqrt{1 - 2\zeta^2} \quad \text{for} \quad 0 \leq \zeta \leq 0.707$$

As the damping ratio $\zeta$ approaches zero, the resonant frequency approaches $w_n$. For $0 \leq \zeta \leq 0.707$, the resonant frequency $w_r$, is less than the damped natural frequency $w_d = w_n \sqrt{1 - \zeta^2}$, which is exhibited in the transient response. From the equation of $w_r$ above it can be seen that for $\zeta > 0.707$, there is no resonant peak.

The magnitude $|G(jw)|$ decrease monotonically with increasing frequency $w$. This magnitude is less than 0 dB for all values of $w > 0$.

The magnitude of the resonant peak $M$, can be found by substituting equation of $w_r$ into equation of $|G(jw)|$ for $0 \leq \zeta \leq 0.707$,

$$M_r = |G(jw)|_{\text{max}} = |G(jw_r)| = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$

For $\zeta > 0.707$ $M_r = 1$.

As $\zeta$ approaches zero, $M_r$ approaches infinity. This means that if the undamped system is excited at its natural frequency, the magnitude of $G(jw)$ becomes infinity.

The phase angle of $G(jw)$ at the frequency where the resonant peak occurs can be obtained by

$$\left|G(jw)\right| = -\tan^{-1} \frac{\sqrt{1 - 2\zeta^2}}{\zeta} = -90^\circ + \sin^{-1} \frac{\zeta}{\sqrt{1 - \zeta^2}}$$

Notes:

a. If gain crossover frequency is less than the phase crossover frequency, the system is stable.

b. If the phase crossover frequency is less than the gain crossover frequency, the system is unstable.